Problem Set 1

1. Find points of local minima of a function $f(x) = 3x^4 - 4x^3 - 60x^2 - 96x + \sqrt{17}$. If there are several of them, indicate the largest one, and if there are none, write 2022 in the answer. (N.B. You need to indicate the x-coordinate.)

   Answer: 4.

   Solution. The derivative is $f'(x) = 12x^3 - 12x^2 - 120x - 96 = 12(x-4)(x+1)(x+2)$. It turns to zero at points $x = 4$, $x = -1$, $x = -2$. Of these points, $x = -1$ is a local maximum, and the other two points are local minima. (If the derivative is negative to the left of the point and positive to the right of the point then it is a minimum. If the derivative is positive to the left of the point and negative to the right of the point then it is a maximum.)

2. Find the value of the expression $\left(\sqrt{\frac{8}{5}} + \sqrt{\frac{1}{2}}\right)\sqrt{3-\sqrt{5}} + \left(\sqrt{\frac{8}{5}} - \sqrt{\frac{1}{2}}\right)\sqrt{3+\sqrt{5}}$.

   Answer: 3.

   Solution. As $3 - \sqrt{5} = \frac{1}{2}\left(6 - 2\sqrt{5}\right) = \frac{1}{2}\left(\sqrt{5}-1\right)^2$, we get that

$$\left(\sqrt{\frac{8}{5}} + \sqrt{\frac{1}{2}}\right)\sqrt{3-\sqrt{5}} = \left(\sqrt{\frac{8}{5}} + \sqrt{\frac{1}{2}}\right)\frac{\sqrt{5}-1}{\sqrt{2}}.$$

   Similarly,

$$\left(\sqrt{\frac{8}{5}} - \sqrt{\frac{1}{2}}\right)\sqrt{3+\sqrt{5}} = \left(\sqrt{\frac{8}{5}} - \sqrt{\frac{1}{2}}\right)\frac{\sqrt{5}+1}{\sqrt{2}}.$$

   Hence, the whole expression is equal to

$$\frac{1}{2\sqrt{5}}\left(\left(4+\sqrt{5}\right)\left(\sqrt{5}-1\right) + \left(4-\sqrt{5}\right)\left(\sqrt{5}+1\right)\right) = \frac{1}{2\sqrt{5}}\left(1 + 3\sqrt{5} + 3\sqrt{5} - 1\right) = 3.$$

3. x_1 and x_2 are roots of the equation $5x^2 - 3x - 10 = 0$. Find $\frac{x_1}{x_2^2} + \frac{x_2}{x_1^2}$.

   Answer: 0.954.

   Solution. The discriminant of the equation is equal to $D = 3^2 + 4\cdot 5\cdot 10 = 209 > 0$, and so the equation has two distinct roots x_1 and x_2. According to Vieta's formulae, $x_1 + x_2 = \frac{3}{5}$, $x_2 x_2 = \frac{-10}{5} = -2$. Thus

$$\frac{x_1}{x_2^2} + \frac{x_2}{x_1^2} = \frac{(x_1+x_2)^3 - 3x_1x_2(x_1+x_2)}{(x_1x_2)^2} = \frac{(3/5)^3 - 3(-2)(3/5)}{(-2)^2} = 0.954.$$

4. Solve the system of inequalities

$$\begin{cases} \log_{6-0.2x}(40x^2 - x^3 - 300x) \geq 1 + \log_{6-0.2x}(25x), \\ \frac{x^2-20x-25}{x-20} + \frac{75}{x-30} \leq x. \end{cases}$$

   As the answer, indicate the number of its integer solutions. If there are infinitely many of them, write 2022 instead.

   Answer: 5.

   Solution. Let us consider the first inequality. Its domain is given by

$$\begin{cases} 6 - 0.2x > 0, \\ 6 - 0.2x \neq 1, \\ 40x^2 - x^3 - 300x > 0, \\ 25x > 0 \end{cases} \Leftrightarrow \begin{cases} x < 30, \\ x \neq 25, \\ x \in (-\infty; 0) \cup (10; 30), \\ x > 0 \end{cases} \Leftrightarrow x \in (10; 25) \cup (25; 30).$$

Applying rationalisation method (inequalities $\log_a b \geq \log_a c$ and $(a-1)(b-c) \geq 0$ are equivalent on their domains), we have that

$$\log_{6-0.2x}\left(40x^2 - x^3 - 300x\right) \geq 1 + \log_{6-0.2x}(25x) \Leftrightarrow$$
$$\Leftrightarrow \log_{6-0.2x}\left(40x^2 - x^3 - 300x\right) \geq \log_{6-0.2x}\left(150x - 5x^2\right) \Leftrightarrow$$
$$\Leftrightarrow (6 - 0.2x - 1)\left(40x^2 - x^3 - 300x - 150x + 5x^2\right) \geq 0 \Leftrightarrow x(x-25)(x-15)(x-30) \geq 0.$$

Therefore, $x \in (-\infty; 0] \cup [15; 25] \cup [30; +\infty)$, and taking the domain into account, we have $15 \leq x < 25$.

Let us solve the second inequality:

$$\frac{x^2 - 20x}{x-20} - \frac{25}{x-20} + \frac{75}{x-30} \leq x \Leftrightarrow \frac{-25}{x-20} + \frac{75}{x-30} \leq x \Leftrightarrow$$
$$\Leftrightarrow \frac{x-15}{(x-20)(x-30)} \leq 0 \Leftrightarrow x \in (-\infty; 15] \cup (20; 30).$$

Intersecting two sets obtained above, we get $x \in \{15\} \cup (20; 25)$. There are exactly five integers in the set.

5. Two yachts move along straight lines and uniformly towards a small island. At the initial moment the positions of the yachts and the island form an equilateral triangle. After the first vessel has passed 40 kilometres, the triangle indicated above becomes a right one. At the moment the second yacht has reached the island, the first yacht has 60 kilometres yet to go. Find the distance between the yachts at the initial moment.

   Answer: 120.

   Solution. Let us denote the side of the initial equilateral triangle as x. Apparently, the first yacht moves at a slower speed as it reaches the port later. Let A and B be positions of the yachts at the start; A_1 and B_1 positions of the yachts when the first of them has passed 40 kilometres; and let C be the position of the port.

   Triangle A_1B_1C is a right triangle, and $\angle A_1B_1C = 90°$, $\angle A_1CB_1 = 60°$. Therefore, $A_1C = x - 40$, $B_1C = \frac{1}{2}A_1C = \frac{x}{2} - 20$, $BB_1 = x - B_1C = \frac{x}{2} + 20$.

   Now we can conclude that for going 40 kilometres the first yacht need as much time as the second one needs for going $\frac{x}{2} + 20$ kilometres. On the other hand, the first and the second yachts need the same time to travel $x - 60$ and x kilometres respectively. Hence, the ratio of speeds can be expressed in two different ways: $\frac{v_1}{v_2} = \frac{x-60}{x} = \frac{40}{0.5x+20}$. Simplifying the equation, we have $x^2 - 100x - 2400 = 0$, and so $x = 120$ or $x = -20$. Since x is a positive number, $x = 120$.

6. Circles Ω and ω intersect at points P and Q. Line ℓ intersects with Ω at points A and C, and with ω at points B and D. It is known that B lies between A and C; C lies between B and D. PQ and AD intersect at point F. Find BF if it is known that $BC = 8$, $AB : CD = 3 : 2$.

   Answer: 4.8.

   Solution. Let us denote $AB = 3y$, $BF = x$. It follows from the task that $CF = 8 - x$, $CD = 2y$. According to the intersecting chords theorem, $AF \cdot CF = PF \cdot QF$ and $PF \cdot QF = BF \cdot DF$. It implies that $AF \cdot CF = BF \cdot DF$, or using the designations introduced above, $(3y + x)(8 - x) = x(8 - x + 2y)$. Simplifying we obtain $24y = 5xy$, and so $x = 4.8$.

7. Numbers a, b, c satisfy the system of equations

$$\begin{cases} 6ac + 3a = 2c - 2, \\ ab + bc = 2(c - a + 1), \\ bc - 6ac + b = 3a + 3. \end{cases}$$

What is the largest possible value of the expression $6a + b - c$?

Answer: 10.5.

Solution. Adding the first and the third equations, we get $bc + b = 2c + 1$, $b(c + 1) = 2c + 1$. It is easy to see that $c = -1$ does not satisfy the equation, and so $c + 1 \neq 0$. Therefore, $b = \frac{2c+1}{c+1}$. In the same way, we can express a from the first equation: $a = \frac{2c-2}{6c+3}$. Substituting the expressions into the second equation of the initial system yields

$$\frac{(2c+1)(2c-2)}{(c+1)(6c+3)} + \frac{2c^2+c}{c+1} = 2\left(c - \frac{2c-2}{6c+3} + 1\right).$$

After multiplying both sides by the common denominator, expanding and combining like terms, we have $10c^2 + 23c + 12 = 0$, and so $c = -\frac{3}{2}$ or $c = -\frac{4}{5}$. Computing the corresponding values of a and b we finally have that the system has two solutions $\left(\frac{5}{6}; 4; -\frac{3}{2}\right)$ and $\left(2; -3; -\frac{4}{5}\right)$. The largest value of $6a + b - c$ is equal to 10.5.

8. Solve the equation $1 + \sin(4x) - \cos(4x) = 2\sin(5x)\cos x$. As the answer, indicate the number of its roots on an interval $-4\pi \leq x \leq 7\pi$.

Answer: 45.

Solution. The equation is equivalent to each of the following equations:

$$1 + \sin 4x - \cos 4x = 2\sin 5x\cos x \Leftrightarrow 1 - \cos 4x = \sin 6x \Leftrightarrow$$
$$\Leftrightarrow 2\sin^2 2x = 3\sin 2x - 4\sin^3 2x \Leftrightarrow \sin 2x\left(4\sin^2 2x + 2\sin 2x - 3\right) = 0;$$

therefore $\sin 2x = 0$ or $\sin 2x = \frac{\sqrt{13}-1}{4}$ (the third root $\sin 2x = -\frac{\sqrt{13}+1}{4}$ is not suitable since it is less than -1). Solving the equations obtained above, we have $x = \frac{\pi k}{2}$, $x = \frac{1}{2}\arcsin\frac{\sqrt{13}-1}{4} + \pi k$, $x = \frac{\pi}{2} - \frac{1}{2}\arcsin\frac{\sqrt{13}-1}{4} + \pi k$, $k \in \mathbb{Z}$. On the interval $-4\pi \leq x \leq 7\pi$ there are 23 roots in the first series $(-8 \leq k \leq 14)$, 11 roots in the second $(-4 \leq k \leq 6)$, and 11 roots in the third $(-4 \leq k \leq 6)$ – there are 45 roots in total.

Due to the error in the task 8 we credited only two answers options to the question, the one which is actually right answer and the one which is right to the question with an error.

Problem Set 2.

1. Find the value of the expression x^3-6x if $x=\sqrt[3]{\sqrt{353}-19}-\sqrt[3]{\sqrt{353}+19}$.

   Answer: -38.

   Solution.

$$x^3-6x=\left(\sqrt{353}-19\right)-3\sqrt[3]{\left(\sqrt{353}-19\right)^2\left(\sqrt{353}+19\right)}+3\sqrt[3]{\left(\sqrt{353}-19\right)\left(\sqrt{353}+19\right)^2}-$$
$$-\left(\sqrt{353}+19\right)-6\left(\sqrt[3]{\sqrt{353}-19}-\sqrt[3]{\sqrt{353}+19}\right)=$$
$$=-38-3\sqrt[3]{\left(\sqrt{353}-19\right)\left(\sqrt{353}+19\right)}\left(\sqrt[3]{\sqrt{353}-19}-\sqrt[3]{\sqrt{353}+19}\right)-$$
$$-6\left(\sqrt[3]{\sqrt{353}-19}-\sqrt[3]{\sqrt{353}+19}\right)=-38-3\sqrt[3]{353-19^2}\left(\sqrt[3]{\sqrt{353}-19}-\sqrt[3]{\sqrt{353}+19}\right)-$$
$$-6\left(\sqrt[3]{\sqrt{353}-19}-\sqrt[3]{\sqrt{353}+19}\right)=-38.$$

2. Let $y=ax+b$ be an equation of a tangent line to the graph $y=x^3-7x^2$ that passes through point $D(1;-6)$ with the least possible slope. Find the value of $3a+2b$.

   Answer: -27.

   Solution. Let x_0 be the point of tangency. Then we have $y(x_0)=x_0^3-7x_0^2$, $y'(x_0)=3x_0^2-14x_0$, and the equation of the tangent line is $y=(3x_0^2-14x_0)(x-x_0)+x_0^3-7x_0^2$. Since the line passes through point $D(1;-6)$ we substitute the coordinates of D into the equation, and we get $-6=(3x_0^2-14x_0)(1-x_0)+x_0^3-7x_0^2$, $x_0^3-5x_0^2+7x_0-3=0$. Factorising the left side we obtain $(x_0-1)^2(x_0-3)=0$. If $x_0=1$ the tangent line is $y=-11x+5$, and if $x_0=3$ we have $y=-15x+9$. The second line has the smallest value of slope, and so $a=-15$, $b=9$, $3a+2b=-27$.

3. Find the largest root of the equation $\sqrt{12x^2-63x-29}=7-2x$. If it has no roots write 2022 in the answer.

   Answer: -1.625.

   Solution. The equation is equivalent to the following

$$\begin{cases}12x^2-63x-29=(7-2x)^2,\\ 7-2x\geq 0\end{cases}\Leftrightarrow\begin{cases}8x^2-35x-78=0,\\ x\leq 3.5\end{cases}\Leftrightarrow\begin{cases}\left[\begin{array}{l}x=6,\\ x=-\frac{13}{8},\end{array}\right.\\ x\leq 3.5\end{cases}\Leftrightarrow x=-\frac{13}{8}.$$

   Representing it as a decimal fraction we obtain the final result $x=-1.625$.

4. Solve the system of inequalities

$$\begin{cases}2\log_2\frac{x-10}{x+12}+\log_2(x+12)^2\geq 2+\log_2 100,\\ 4^{0.1x+1}-33\cdot\sqrt[10]{2^x}+8\leq 0.\end{cases}$$

   As the answer, indicate the number of its integer solutions. If there are infinitely many of them, write 2022 instead.

   Answer: 9.

Solution. Let us consider the first inequality. Its domain is given by $\frac{x-10}{x+12} > 0$, $(x+12)^2 > 0$, which yields $x \in (-\infty; -12) \cup (10; +\infty)$. On the domain the inequality is equivalent to the following

$$\log_2 \left(\frac{x-10}{x+12}\right)^2 + \log_2 (x+12)^2 \geq 2 + \log_2 100 \Leftrightarrow \log_2 (x-10)^2 \geq \log_2 400 \Leftrightarrow$$
$$\Leftrightarrow (x-10)^2 \geq 400 \Leftrightarrow (x+10)(x-30) \geq 0 \Leftrightarrow x \in (-\infty; -10] \cup [30; +\infty).$$

And taking the domain into account, we finally have $x \in (-\infty; -12) \cup [30; +\infty)$.

In the second inequality we introduce a substitution $4^{0.1x} = t$ which yields $4t^2 - 33t + 8 \leq 0$, $(t-8)(4t-1) \leq 0$, $\frac{1}{4} \leq t \leq 8$. Consequently, $\frac{1}{4} \leq 2^{0.1x} \leq 8$, $-2 \leq \frac{x}{10} \leq 3$, $-20 \leq x \leq 30$.

Intersecting the solution sets of the inequalities, we obtain $x \in [-20; -12) \cup \{30\}$. This set has exactly 9 integers in it.

5. Three pedestrians had to go from A to B along the same road. The first and the second left A at 7:00 in the morning, and the third one left two hours later. All pedestrians move at a constant speed. It is known that by noon, none of the pedestrians overcame another ones. The distance between the first and the third pedestrians at noon was 4 times less than at 9:00. The distance between the second and the third pedestrians at 9:00 was $\frac{10}{7}$ greater than at noon. Find the ratio of the speed of the first pedestrian to the speed of the second pedestrian.

Answer: 0.8.

Solution. Let v_1, v_2, v_3 be the pedestrians' speeds. At 9:00 the positions of the pedestrians are $2v_1$, $2v_2$, 0, and at 12:00 they are $5v_1$, $5v_2$, $3v_3$. Therefore, the distances between the first and the third pedestrians at 9:00 and at 12:00 are $2v_1$ and $5v_1 - 3v_3$. According to the task, $4(5v_1 - 3v_3) = 2v_1$. Similarly, $2v_2 = \frac{10}{7}(5v_2 - 3v_3)$. The first equation yields $v_1 = \frac{2}{3}v_3$, and the second yields $v_2 = \frac{5}{6}v_3$. Consequently, $\frac{v_1}{v_2} = \frac{2}{3} \cdot \frac{6}{5} = 0.8$.

6. Circles Ω (centred at A) and ω (centred at B) are tangent to line ℓ at points P and Q respectively. Line segment AB intersects with line ℓ. It is known that the radii of the circles are equal to 16 and 5, and $PQ : AB = 20 : 29$. Find AB.

Answer: 29.

Solution. Let us drop perpendicular AH from point A onto line BQ. Quadrilateral $APQH$ is a rectangle, and so $BH = BQ + QH = BQ + AP = 21$. Let $PQ = 20x$. Then $AH = PQ = 20x$, $AB = 29x$. Using Pythagorean theorem for a right triangle ABH we obtain $(29x)^2 = (21x)^2 + 20^2$. From here $x = 1$, hence $AB = 29$.

7. Numbers a, b satisfy the system of equations

$$\begin{cases} a^2 + ab - 2b^2 + 8a + 10b + 12 = 0, \\ a^2 + 3ab + 2b^2 - a + b - 6 = 0. \end{cases}$$

What is the largest possible value of the expression $b^2 - a$?

Answer: 12.

Solution. The first equation of the system can be written as $a^2 + (b+8)a - 2b^2 + 10b + 12 = 0$. It is a quadratic equation with respect to a. Its discriminant is $D = (b+8)^2 + 4(2b^2 - 10b - 12) = (3b-4)^2$, and the roots are $a = \frac{-b-8\pm(3b-4)}{2}$, so $a = -2b - 2$ or $a = b - 6$. Now we substitute the values into the second equation.

If $a = -2b - 2$ we have $4b^2 + 8b + 4 - 6b^2 - 6b + 2b^2 + 2b + 2 + b - 6 = 0$; therefore, $b = 0$ and $a = -2$. If $a = b - 6$ then $b^2 - 12b + 36 + 3b^2 - 18b + 2b^2 - b + 6 + b - 6 = 0$, $b^2 - 5b + 6 = 0$, and we have two more solutions $b = 3$, $a = -3$ and $b = 2$, $a = -4$. The maximum value of $b^2 - a$ is reached for $(-3; 3)$ and is equal to 12.

8. $KLMN$ is a rhombus, and $\angle KLM = \arccos \frac{8}{17}$, $KN = 34$. Altitudes KA and KB are dropped from point K onto sides LM and MN. Find the radius of a circle inscribed into quadrilateral $KAMB$. If you think that it is impossible to inscribe a circle into $KAMB$, write 2022 as the answer.

   Answer: 11.25.

   Solution. Let us consider a right triangle AKL. From it we find that $AL = KL \cos \angle ALK = 34 \cdot \frac{8}{17} = 16$; $AK = \sqrt{KL^2 - AL^2} = 30$. Besides that, $AM = LM - AL = 18$. Similarly, $BK = 30$, $BM = 18$.

   Let us consider quadrilateral $AKBM$. Angles A and B are right, and so it is convex; $AM + BK = AK + BM$. Therefore, $KAMB$ is a tangential quadrilateral. Due to symmetry, the centre of its incircle O is situated on its diagonal KM. Let us drop perpendiculars OE and OF from point O onto sides AK and AM respectively. Let r be the radius of the incircle; then $AE = r$, $EK = 30 - r$. Triangles OEK and MAK are similar, and so $\frac{OE}{AM} = \frac{KE}{AE}$, i.e. $\frac{r}{18} = \frac{30-r}{30}$. Therefore, $r = \frac{45}{4} = 11.25$.

Problem Set 3.

1. Solve the equation $\sqrt{2x^2+3x+2}-\sqrt{2x^2+3x-5}=1$. As the answer, indicate its smallest root. If the equation has no roots write 2022.

   Answer: -3.5.

   Solution. Let us denote $2x^2+3x=t$. Then we have

$$\sqrt{t+2}=\sqrt{t-5}+1 \Leftrightarrow \begin{cases} t+2=t-4+2\sqrt{t-5}, \\ t\geq 5 \end{cases} \Leftrightarrow \begin{cases} \sqrt{t-5}=3, \\ t\geq 5 \end{cases} \Leftrightarrow t=14.$$

   Therefore, $2x^2+3x=14$, $x=2$ or $x=-\frac{7}{2}$. The smallest root is $x=-\frac{7}{2}=-3.5$.

2. Line $y=ax+b$ passes through point $(4;1)$ and is tangent to the graph of a function $y=x^2-4x+2$. What is the largest possible value of a?

   Answer: 6.

   Solution. If line $y=ax+b$ passes through point $(4;1)$ then $1=4a+b$, and so $b=1-4a$, and the equation of the line can be written as $y=ax+1-4a$. The line is tangent to the parabola $y=x^2-4x+2$ if and only if the system of equations $y=ax+1-4a$ and $y=x^2-4x+2$ has exactly one solution. Equating the right sides yields $x^2-(4+a)x+(4a+1)=0$. The discriminant has to be equal to zero, so we have $(4+a)^2-4(4a+1)=0$, $a=2$ or $a=6$. The largest possible value of a is 6.

3. Find the largest negative root of the equation $\cos(x\pi)+\frac{1}{\cos(x\pi)}=\cos^2(x\pi)+\frac{1}{\cos^2(x\pi)}$.

   Answer: -2.

   Solution. Let us denote $t=\cos(x\pi)+\frac{1}{\cos(x\pi)}$. Raising both sides to the second power, we have $t^2=\cos^2(x\pi)+2+\frac{1}{\cos^2(x\pi)}$. The initial equation becomes $t=t^2-2$, so $t=1$ or $t=2$.

   If $t=-1$ we get an equation $\cos(x\pi)+\frac{1}{\cos(x\pi)}=-1$, $\cos^2(x\pi)+\cos(x\pi)+1=0$ that has no solutions.

   If $t=2$ we have $\cos(x\pi)+\frac{1}{\cos(x\pi)}=2$, $\cos^2(x\pi)-2\cos(x\pi)+1=0$, $\cos(x\pi)=1$, $x\pi=2\pi k$, $k\in\mathbb{Z}$. Finally, $x=2k$, and the largest negative root is equal to -2.

4. Solve the inequality

$$\frac{4\log_2(x+3)-11}{\log_2^2(x+3)-5\log_2(x+3)+6}\geq 1+\log_{0.125x+0.375}2.$$

   As the answer, indicate the number of its integer solutions. If there are infinitely many of them, write 2022 instead.

   Answer: 27.

   Solution. Let $t=\log_2(x+3)$. Then we have $\log_{0.125x+0.375}2=\frac{1}{\log_2\left(\frac{x+3}{8}\right)}=\frac{1}{\log_2(x+3)-3}=\frac{1}{t-3}$, and the inequality can be written as

$$\frac{4t-11}{(t-3)(t-2)}-1-\frac{1}{t-3}\geq 0 \Leftrightarrow \frac{-t^2+8t-15}{(t-3)(t-2)}\geq 0 \Leftrightarrow \frac{(t-5)(t-3)}{(t-3)(t-2)}\leq 0 \Leftrightarrow \left[\begin{array}{l} 2<t<3, \\ 3<t\leq 5. \end{array}\right.$$

   Returning to variable x we find

$$\left[\begin{array}{l} 2<\log_2(x+3)<3, \\ 3<\log_2(x+3)\leq 5 \end{array}\right. \Leftrightarrow \left[\begin{array}{l} 4<x+3<8, \\ 8<x+3\leq 32 \end{array}\right. \Leftrightarrow x\in(1;5)\cup(5;29].$$

   The number of integers on the solution set is equal to 27.

5. Numbers a, b satisfy the system of equations

$$\begin{cases} \frac{a^3}{b} - 2ab = 16, \\ \frac{b^3}{2a} + 3ab = 25. \end{cases}$$

What is the least possible value of the expression $2a + b$?

Answer: -10.

Solution. Moving the terms $2ab$ and $3ab$ to the right sides of the equations and multiplying the equations obtained, we have $\frac{a^3}{b} \cdot \frac{b^3}{2a} = (16 + 2ab)(25 - 3ab)$, $\frac{13}{2}(ab)^2 - 2(ab) - 400$. Solving the quadratic equation with respect to ab we have $ab = -\frac{100}{13}$ or $ab = 8$. Now we can express b and substitute it into the first equation of the initial system.

If $b = \frac{8}{a}$ then $\frac{a^4}{8} = 16 + 16$, $a^4 = 256$, $a = \pm 4$, and we have two solutions $(4; 2)$ and $(-4; -2)$.

If $b = -\frac{100}{13a}$ we have $-\frac{13a^4}{100} = 16 - \frac{200}{13}$, $a^4 = -\frac{800}{169}$, and there are no solutions.

The smallest possible value of $2a + b$ is equal to -10.

6. Find the minimum value of a such that equations $3ax^2 - 5x + 2a = 0$ and $2x^2 + ax - 3 = 0$ have a common root.

Answer: -1.

Solution. Let x_0 be the common root of the equations. Then we get that numbers a and x_0 satisfy a system of equations $3ax_0^2 - 5x_0 + 2a = 0$, $2x_0^2 + ax_0 - 3 = 0$. Expressing a from the first equation yields $a = \frac{5x_0}{3x_0^2+2}$. Substituting it into the second equation, we obtain $2x_0^2 + \frac{5x_0^2}{3x_0^2+2} - 3 = 0$, hence $x_0^4 = 1$ and $x_0 = \pm 1$. If $x_0 = 1$ then $a = 1$, and if $x_0 = -1$ then $a = -1$. The minimum value of a is equal to -1.

7. The diagonals of a parallelogram are equal to 22 and 13, and one of its angles is equal to $\arcsin \frac{20}{29}$. Find the area of the parallelogram.

Answer: 75.

Solution. Let the sides of the parallelogram be equal to x and y, and let us denote $\arcsin \frac{20}{29} = \psi$. Then $\cos\psi = \sqrt{1 - \left(\frac{20}{29}\right)^2} = \frac{21}{29}$. Using law of cosines two times, we get $x^2 + y^2 - 2xy \cdot \frac{21}{29} = 13^2$, $x^2 + y^2 - 2xy \cdot \left(-\frac{21}{29}\right) = 2^2$. Subtracting the first equation from the second one we have $\frac{84}{49}xy = 315$. The area of the parallelogram is equal to $xy \sin\psi = \frac{29}{84} \cdot 315 \cdot \frac{20}{29} = 75$.

8. X and Y are three-digit numbers. If X is written before Y, the six-digit number obtained is divisible by Y, and the quotient is equal to 601. If Y is written before X and the number obtained is divided by X, the quotient is equal to 1667, and the residue to 306. Find Y.

Answer: 765.

Solution. It follows from the task that $1000x + Y = 601Y$, $1000Y + X = 1667X + 306$. Expressing Y from the first equation, we have $Y = \frac{5X}{3}$. We substitute it into the second equation, and we obtain $\frac{5000X}{3} = 1666X + 306$, $X = 459$, and $Y = 765$.

Problem Set 4.

1. The sum of the tenth, the twenty-eighth and the thirty-first terms of an arithmetic sequence is equal to (-19). Find the sum of the first forty-five terms of the sequence.

   Answer: -285.

   Solution. Let us denote the terms of the sequence as $a_1, a_2, \ldots$, and its common denominator as d. It follows from the task that $a_{10} + a_{28} + a_{31} = -19$, $a_1 + 9d + a_1 + 27d + a_1 + 30d = -19$, $3a_1 + 66d = -19$. Now we can compute the sum in question: $a_1 + a_2 + \ldots + a_{45} = \frac{a_1+a_{45}}{2} \cdot 45 = \frac{a_1+a_1+44d}{2} \cdot 45 = 15\,(3a_1 + 66d) = 15 \cdot (-19) = -285$.

2. The bases of the trapezoid are equal to 14 and 29, and its lateral sides are 26 and 37. Find the area of the trapezoid.

   Answer: 447.2.

   Solution. Let $ABCD$ be the given trapezoid, and $AD = 14$, $CD = 37$, $BC = 29$, $AB = 26$. Let us draw a line that passes through point D and is parallel to AB and let M be the intersection point of this line and BC. Quadrilateral $ABMD$ is a parallelogram, thus $DM = AB = 26$, $CM = BC - BM = 29 - 14 = 15$. We know all sides in triangle CDM: $CM = 15$, $CD = 37$, $DM = 26$. The semi-perimeter of the triangle is 39, and its area is $A = \sqrt{39(39-15)(39-37)(39-26)} = 156$. Therefore, the altitude h of the triangle dropped from point D is equal to $h = \frac{2A}{CM} = \frac{2\cdot 156}{15} = \frac{104}{5}$. Now we can find the area of the trapezoid. It equals $\frac{AD+BC}{2}h = \frac{14+29}{2} \cdot \frac{104}{5} = 447.2$.

3. Find all values of parameter a such that the equation $9^x + (1+a)\cdot 3^x + 3 - 2a = 0$ has exactly one solution. As the answer, indicate the smallest natural value of a. If no natural values of a satisfy the condition above, write 2022 instead.

   Answer: 2.

   Solution. The equation is quadratic with respect to $t = 3^x$. Its discriminant D is equal to $(1+a)^2 - 4(3-2a) = a^2 + 10a - 11$.

   If $D < 0$ the equation has no solutions. If $D = 0$ there are two possibilities. $a = 1$ yields an equation $9^x + 2\cdot 3^x + 1 = 0$, which has no solutions. If $a = -11$ we have $9^x - 10\cdot 3^x + 25 = 0 \Leftrightarrow 3^x = 5$, i.e. there is exactly one solution.

   If $D > 0$ (which happens for $a < -11$ or $a > 1$) there are two different values t_1 and t_2 that satisfy the equation. If we recall that we are interested in natural values of parameter a, we only need to consider $a \geq 2$. For the initial equation to have exactly one solution x it is necessary and sufficient that $t_1 \leq 0$ and $t_2 > 0$. But according to Vieta's formulae, $x_1 x_2 = 3 - 2a < 0$ if $a \geq 2$. It means that all values $a \geq 2$ satisfy the task, and the smallest of them is $a = 2$.

4. Solve the inequality

$$\frac{\log_5^2 x + \log_5 x - 4}{\log_5(0.2x)} + \frac{8\log_5^2 x - 32\log_5 x + 5}{\log_5(x/625)} \leq 9\log_5 x + 2.$$

   As the answer, indicate sum of lengths of all intervals obtained. If there is at least one infinite interval, write 2022 instead.

   Answer: 620.2.

Solution. Let $\log_5 x = t$. Then we have

$$\frac{t^2+t-4}{t-1}+\frac{8t^2-32t+5}{t-4}\leq 9t+2 \Leftrightarrow$$
$$\Leftrightarrow \frac{(t^2+t-4)(t-4)+(8t^2-32t+5)(t-1)-(9t+2)(t^2-5t+4)}{(t-1)(t-4)}\leq 0 \Leftrightarrow$$
$$\Leftrightarrow \frac{3t+3}{(t-1)(t-4)}\leq 0 \Leftrightarrow \left[\begin{array}{l} t\leq -1, \\ 1<t<4. \end{array}\right.$$

We proceed finding the values of x:

$$\left[\begin{array}{l} \log_5 x\leq -1, \\ 1<\log_5 x<4 \end{array}\right. \Leftrightarrow \left[\begin{array}{l} 0<x\leq \frac{1}{5}, \\ 5<x<625. \end{array}\right.$$

The sum of lengths of the intervals is equal to $\left(\frac{1}{5}-0\right)+(625-5)=620.2$.

5. Numbers a, b satisfy the system of equations

$$\begin{cases} 9a^4+4a^2b+3=0, \\ 9a^2b^2+4b^3+27=0. \end{cases}$$

What is the least possible value of the expression $a+3b$?

Answer: -10.

Solution. Let us subtract the second equation from the first equation multiplied by 9. It yields $81a^4+36a^2b-9a^2b^2-4b^3=0$, and factorising the left side we obtain $9a^2\left(9a^2-b^2\right)+4b\left(9a^2-b^2\right)=0$, $\left(9a^2+4b\right)(3a-b)(3a+b)=0$. Therefore, we have three possibilities: $9a^2+4b=0$ or $3a-b=0$ or $3a+b=0$. We express b and substitute into the first equation of the initial system.

If $b=-\frac{9a^2}{4}$ then $3=0$, and this case yields no solutions.

If $b=3a$ we have

$$9a^4+12a^3+3=0 \Leftrightarrow \left(3a^4+3a^3\right)+\left(a^3+1\right)=0 \Leftrightarrow 3a^3(a+1)+(a+1)\left(a^2-a+1\right)=0 \Leftrightarrow$$
$$\Leftrightarrow (a+1)\left(3a^3+a^2-a+1\right)=0 \Leftrightarrow (a+1)\left(\left(a^3+a^2\right)+\left(a^3-a\right)+\left(a^3+1\right)\right)=0 \Leftrightarrow$$
$$\Leftrightarrow (a+1)^2\left(a^2+a^2-a+a^2-a+1\right)=0 \Leftrightarrow a=-1.$$

Consequently, we have a solution $(-1;-3)$.

If $b=-3a$ we get $9a^4-12a^3+3 \Leftrightarrow (a-1)^2\left(9a^2+6a+3\right)=0 \Leftrightarrow a=1$. We get one more solution $(1;-3)$.

Thus the least value of $a+3b$ is equal to -10.

6. Quadrilateral $PQRS$ is inscribed into a circle of radius 32.5. It is known that $PS:SR=16:63$, $\angle PQR=90^\circ$, and the perimeter of $PQRS$ is equal to 168. Find the area of the quadrilateral.

Answer: 1428.

Solution. As $\angle PQR=90^\circ$, PR is a diameter of a circle, and $PR=2\cdot 32.5=65$. Let $PS=16x$, then $RS=63x$. Using Pythagorean theorem for triangle PRS we get $(16x)^2+(63x)^2=65^2$, and so $x=1$, $PS=16$, $RS=63$. Since the perimeter of the quadrilateral is equal to 168, we get that $PQ+QR=168-16-63=89$. If we denote $PQ=y$ we have $QR=89-y$, and due to Pythagoreans theorem $y^2+(89-y)^2=65^2$, $y^2-89y+1848=0$, and so $y=33$ or $y=56$. It means that one of the sides PQ and QR is equal to 33 and another one to 56. The area of $PQRS$ is equal to the sum of areas of two right triangles PQR and PSR, i.e. $\frac{56\cdot 33}{2}+\frac{16\cdot 63}{2}=1428$.

7. Solve the equation $\sin 2x + 2\sin 4x = \tan 6x \cos 2x$. As the answer, indicate the number of its roots on an interval $10\pi < x < 40\pi$.

   Answer: 239.

   Solution. The equation is equivalent to the following:

$$\begin{cases}(\sin 2x + 2\sin 4x)\cos 6x = \sin 6x\cos 2x,\\ \cos 6x \neq 0\end{cases} \Leftrightarrow \begin{cases}2\sin 4x\cos 6x = \sin 6x\cos 2x - \cos 6x\sin 2x,\\ \cos 6x \neq 0\end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases}2\sin 4x\cos 6x = \sin 4x,\\ \cos 6x \neq 0\end{cases} \Leftrightarrow \begin{cases}\left[\begin{array}{l}\sin 4x = 0,\\ \cos 6x = 0.5,\end{array}\right.\\ \cos 6x \neq 0\end{cases} \Leftrightarrow \left[\begin{array}{l}\begin{cases}x = \frac{\pi k}{4}, k \in \mathbb{Z},\\ \cos 6x \neq 0,\end{cases}\\ x = \pm\dfrac{\pi}{18} + \dfrac{\pi n}{3}, n \in \mathbb{Z}.\end{array}\right.$$

   It is clear that the restriction $\cos 6x \neq 0$ is only relevant for the case when $\sin 4x = 0$. If we consider it, the equation yields $x = \frac{\pi k}{4}$, $k \in \mathbb{Z}$, and so $\cos 6x = \cos\frac{3\pi k}{2}$, which is equal to 0 if k is an odd number, and equal to ± 1 if k is even. Therefore, $k = 2m$, $m \in \mathbb{Z}$, and the roots of the equation we get out of here are $x = \frac{\pi m}{2}$, $m \in \mathbb{Z}$.
   The sets of roots obtained above do not intersect with each other. Indeed, if we equate the roots, we have $\frac{\pi m}{2} = \pm\frac{\pi}{18} + \frac{\pi n}{3} \Leftrightarrow 6m = \frac{2}{3} + 4n$, which is impossible since the left side is an integer and the right side is not.
   To find the number of roots we consider inequalities $10\pi < \frac{\pi m}{2} < 40\pi$ and $10\pi < \pm\frac{\pi}{18} + \frac{\pi n}{3} < 40\pi$. The former yields $20 < m < 80$, and the latter yields $\mp\frac{1}{6} + 30 < n < \mp\frac{1}{6} + 120$. There are 59 different values of m and 180 values of n (90 values for each of the signs). Thus there are $59 + 180 = 239$ integer solutions on the interval $10\pi < x < 40\pi$.

8. k is a natural number. If we add 1000 to k we obtain a square number. If 1532 is added to k we get a square number as well. Find the least possible value of k.

   Answer: 16 424.

   Solution. Let $k+1000l^2$ and $k+1532 = m^2$. Without loss of generality we can assume that both l and m are positive integers. Subtracting the first equation from the second one, we have $532 = m^2 - l^2$, $(m-l)(m+l) = 2^2 \cdot 7 \cdot 19$. Numbers $m+l$ and $m-l$ are either both odd or both even. Since their product is an even number, both of them have to be even numbers. We can also notice that $m+l > m-l$. Hence, there are only two possibilities:

$$\begin{cases}m + l = 38,\\ m - l = 14\end{cases} \quad \text{or} \quad \begin{cases}m + l = 266,\\ m - l = 2.\end{cases}$$

   Solving the systems, we get $m = 26$, $l = 12$ or $m = 134$, $l = 132$. The corresponding values of k are -856 and 16 424. As k is a natural number, it cannot be negative, and so $k = 16\,424$.